Math 113 Fall 09 Exam 3 Key

1. $y' = -\tan x$.

$$s = \int_0^{\pi/4} \sqrt{1+\tan^2 x}\, dx = \int_0^{\pi/4} \sec x\, dx$$

$$= ln\, |\sec x + \tan x|_0^{\pi/4} = ln(\sqrt{2}+1).$$

2. $y' = \frac{x}{2} - \frac{1}{2x}$.

$$s = \int_1^2 \sqrt{1+(\frac{x}{2}-\frac{1}{2x})^2}\, dx = \int_1^2 \sqrt{1+\frac{x^2}{2}-\frac{1}{2}+\frac{1}{4x^2}}\, dx$$

$$= \int_1^2 \sqrt{\frac{x^2}{2}+\frac{1}{2}+\frac{1}{4x^2}}\, d = \int_1^2 \sqrt{(\frac{x}{2}+\frac{1}{2x})^2}\, dx$$

$$= \int_1^2 \frac{x}{2}+\frac{1}{2x}\, dx = (\frac{x^2}{4}+\frac{1}{2}\ln x)|_1^2 = 1+\frac{1}{2}\ln 2 - \frac{1}{4} = \frac{3}{4}+\frac{1}{2}\ln 2.$$

3. The length of the line segment is $l = \sqrt{(7-3)^2+(0-3)^2} = 5$. According to formula 2 on page 533, the surface area is given by $\pi(r_1+r_2)l = \pi\cdot(3+7)\cdot 5 = 50\pi$.

4. $y' = \frac{1}{2}(4-x^2)^{-1/2}\cdot(-2x) = \frac{-x}{\sqrt{4-x^2}}$.

$$S = \int_1^2 2\pi\sqrt{4-x^2}\sqrt{1+\frac{x^2}{4-x^2}}\, dx = \int_1^2 2\pi\sqrt{4-x^2}\sqrt{\frac{4}{4-x^2}}\, dx$$

$$= \int_1^2 2\pi\cdot 2\, dx = 4\pi.$$

5.

$$S = \int_1^2 2\pi x\sqrt{1+\frac{x^2}{4-x^2}}\, dx = 2\pi\int_1^2 \frac{2x}{\sqrt{4-x^2}}\, dx$$

Letting $u = 4-x^2$, then $du = -2x\, dx$, and the above becomes

$$-2\pi\int_3^0 u^{-1/2}\, du = 2\pi\cdot 2u^{1/2}|_0^3 = 4\pi\sqrt{3}.$$

6. D.

7. If $h = 0$ at the top of the water, then

$$F = \int_3^5 \rho g \cdot h \cdot 2\, dh = 62.5h^2|_3^5 = 62.5 \cdot 16 = 1000 \text{ lbs.}$$

Note that you can also easily use Pappus' theorem: the depth of the centroid is 4 feet and the area is 4 square feet. The force is then

$$62.5 \cdot 4 \cdot 4 = 1000 \text{ lbs.}$$

8. If $h = 0$ at the top of the water, then we need to break the problem into two pieces.

Top half: using similar triangles the width across the plate at depth h is $l = 2h$. The force is

$$F_1 = \int_0^{\sqrt{2}} 62.5h(2h)\, dh = 125\frac{h^3}{3}|_0^{\sqrt{2}} = \frac{250\sqrt{2}}{3} \text{ lbs.}$$

Bottom half: Using similar triangles the width across the plate at depth h is $l = 2(2\sqrt{2} - h)$. The force is

$$F_2 = \int_{\sqrt{2}}^{2\sqrt{2}} 62.5h(4\sqrt{2} - 2h)\, dh = \frac{500}{3}\sqrt{2}.$$

The total force is then

$$F = F_1 + F_2 = \frac{250\sqrt{2}}{3} + \frac{500}{3}\sqrt{2} = \frac{750}{3}\sqrt{2} = 250\sqrt{2} \text{ lbs.}$$

Again, we can also use Pappus' theorem: The distance from the centroid to the surface is $\sqrt{2}$ feet. The cross sectional area is 4 square feet.

$$F = 62.5 \cdot \sqrt{2} \cdot 4 = 250\sqrt{2} \text{ lbs.}$$

9. Clearly $\overline{x} = 0$. Also, since the region is a half circle, $A = \frac{\pi}{2}$.

$$M_x = \int_{-1}^{1} \frac{1}{2}(\sqrt{1 - x^2})^2\, dx = \int_{-1}^{1} \frac{1}{2}(1 - x^2)\, dx = \frac{1}{2}(x - \frac{1}{3}x^3)_{-1}^{1} = 2/3.$$

Thus, $\overline{y} = \frac{4}{3\pi}$.

10. From the symmetry, $\overline{y} = 1$. Note that the sum of the moments is

$$M_y = -1 \cdot 4 + 1 \cdot 3 = -1, \ A = 4 + 3 = 7.$$

Thus, $\overline{x} = -1/7$.

11. (a) For every ϵ, there is an M so that if $n > M$ then $|a_n - L| < \epsilon$.

    (b) If $a_n \to L$, then $a_{n+1} \to L$, and

    $$L = 4 - \frac{1}{L},$$

    or

    $$L^2 - 4L + 1 = 0.$$

    Using the quadratic formula, we see that

    $$L = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}.$$

    This problem is similar to problem 69 in 11.1, and it can easily be shown that the sequence is increasing and bounded by 4. Hence, the limit has to be the larger of the two: $2 + \sqrt{3}$.

    (c) $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 = 0$

12. (a) ∞.

    (b) e^2.

    (c) 0.

13. (a) Diverges, p series, $p = 1/2 < 1$.

    (b) Converges, telescoping series.

    $$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

    $$1 = A(n+1) + Bn$$

    $A = 1$, $B = -1$.

    $$s_n = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{n+1} \to 1$$

(c) Geometric

$$\sum_{n=1}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}} - 1\right) = \frac{2}{3}.$$

14. (a) The comparison test doesn't work very well for this problem. Most successful answers used the integral test.

$$\int_2^{\infty} \frac{1}{x\ln^2 x}\,dx = \int_{\ln 2}^{\infty} \frac{1}{u^2}\,du = \frac{1}{\ln 2}$$

if $u = \ln x$. Thus, the integral converges.

(b) Since

$$\frac{1}{n^3+1} < \frac{1}{n^3},$$

the series converges by the comparison test. (The limit comparison test could also be used here).

(c) The easiest thing to do is to use the limit comparison test with $1/n$:

$$\frac{\frac{n^2+3n+1}{n^3+2n^2+n+1}}{\frac{1}{n}} = \frac{n^3+3n^2+n}{n^3+2n^2+n+1} \to 1.$$

Thus, the series diverges.

You can use the comparison test as well, but its more difficult. You can drop the extra terms on the numerator to make things smaller, but **that will not work** with the denominator.

(d)

$$\frac{1}{\sqrt{n^3+2n^2+n+1}} < \frac{1}{\sqrt{n^3}},$$

so by the comparison test, the series converges.

15.

$$\int_k^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_k^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}(-\frac{1}{b} + \frac{1}{k}) = \frac{1}{k}.$$

If we require this to be less than $1/1000$, then $k > 1000$.

16. Using the theorem of Pappus, $V = 2\pi d A$ where d is the distance from centroid to axis of revolution and A is the area. Hence, $12\pi = 2\pi\overline{y} \cdot 4$, so $\overline{y} = 3/2$. Also, $8\pi = 2\pi\overline{x} \cdot 4$, so $\overline{x} = 1$.

17. The obvious example would be the harmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

We know it diverges by the integral test, but $\frac{1}{n} \to 0$.